are given in the third and fourth columns for the inclined, and mP^{-1} and nP^{-1} for the parabolic stamp. All the quantities are computed by the three methods described above. The number 1 on the left corresponds to the case $\alpha = 0.5\pi, \lambda = 2$ and the number 2 to $\alpha = 1.5\pi, \lambda = 1$. The values of the constants C and D are taken from /2/.

Therefore, as in /9/, devoted to a wedge with a clamped lower face, joining of the asymptotic solutions for large and small λ has successfully been established for a wedge lower face is stress-free. The method of reducing the integral equation with a symbol of the type cth to infinite systems of the second kind that enable the accuracy of the asymptotic solutions to be monitored can also be considered effective.

REFERENCES

1. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Non-classical Mixed Problems of Elasticity Theory, Nauka, Moscow, 1974.
2. ALEKSANDROV V.M., Contact problems of an elastic wedge, Inzh. Zh., Mekhan. Tverd. Tela, 2, 1967.
3. ALEKSANDROV V.M., Contact problems for an elastic flat wedge, Contact Problems and their Engineering Applications, NIIMASh, Moscow, 1969.
4. NOBLE B., Application of the Wiener-Hopf Method to Solve Partial Differential Equations, IIL, Moscow, 1962.
5. PRUDNIKOV A.P., BRYCHKOV YU.A. and MARICHEV O.I., Integrals and Series. Elementary Functions. Nauka, Moscow, 1981.
6. LAVRENT'YEV M.A. and SHABAT B.V., Methods of the Theory of Functions of a Complex Variable, Nauka, Moscow, 1973.
7. UFLYAND YA.S., Integral Transforms in Problems of the Theory of Elasticity, Izd. Akad. Nauk SSSR, Moscow-Leningrad, 1963.
8. ALEKSANDROV V.M. and KOPASENKO V.V., Contact problem for an elastic wedge with a rigidly clamped face, Prikl. Mekhan., 4, 7, 1968.
9. ALEKSANDROV V.M., On contact problems for an elastic wedge with one clamped face, Izv. Akad. Nauk ArmSSR, Mekhanika, 21, 2, 1968.

Translated by M.D.F.

PMM U.S.S.R., Vol.52, No.4, pp.511-518, 1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

ASYMPTOTIC INTEGRATION OF NON-LINEAR EQUATIONS OF CYLINDRICAL PANEL VIBRATIONS*

L.S. SRUBSHCHIK, A.M. STOLYAR and V.G. TSIBULIN

Complete asymptotic expansions of the solution of the two-dimensional problem of the non-linear vibrations of a cylindrical panel with free curvilinear boundaries are constructed using the boundary layer method /1, 2/ in the case when the parameter δ, equal to the ratio between the lengths of the clamped and free sides, is fairly small. The principal term of the **expansion** for the deflection function is determined from the known non-linear integrodifferential equation of arch vibrations. The discrepancies in satisfying the boundary conditions on the clamped boundaries turn out to be higher-order infinitesimals in δ and are compensated by boundary layer functions that are determined from linear boundary value problems for a biharmonic operator in a half-strip. Calculations are performed by using finite differences for elastic, elasto-plastic cylindrical panels, arches, and rectangular plates subjected to an instantaneously applied transverse step load, and the limits of applicability are established for a monomial expansion. Questions on passage

*Prikl.Matem.Mekhan.,52,4,657-665,1988

to the limit from the three-dimensional equations of the theory of elasticity to two-dimensional equations in the case of thin domains were investigated in /3-5/ for non-linear problems.

1. Formulation of the problem. The equations of the non-linear vibrations of an elastic rectangular cylindrical panel together with the initial and boundary conditions /6/ can be written in the dimensionless form

$$\Delta_1^2 w + \delta^4 \partial_t^2 w - k\delta^2 \partial_y^2 \Phi = \delta^4 q(x, t) + \delta^2 L(w, \Phi) \tag{1.1}$$

$$\Delta_1^2 \Phi + {}^1/_2 \alpha\delta^2 L(w, w) + \alpha k \delta^2 \partial_y^2 w = 0$$

$$[w, \partial_t w]_{t=0} = 0 \tag{1.2}$$

$$[\partial_x^2 \Phi, \partial_x \partial_y \Phi, \partial_y^2 w + \nu\delta^2 \partial_x^2 w, \partial_y^3 w + (2-\nu)\delta^2 \partial_x^2 \partial_y w]_{y=\pm 1} = 0 \tag{1.3}$$

$$[w, \partial_x^2 w, \delta^2 \partial_x^2 \Phi - \nu \partial_y^2 \Phi, \delta^2 \partial_x^3 \Phi + (2+\nu)\partial_x \partial_y^2 \Phi]_{x=\pm 1} = 0 \tag{1.4}$$

$$\left(\Delta_1 = \partial_y^2 + \delta^2 \partial_x^2, \partial_t = \frac{\partial}{\partial t}, \partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}, \delta = \frac{a_2}{a_1}\right)$$

The dimensional and dimensionless quantities are connected by the formulas

$$x_1 = a_1 x, \quad x_2 = a_2 y, \quad W = a_1 w, \quad F = D\Phi, \quad \tau = ct \tag{1.5}$$

$$c^2 = \rho h a_1^4 D^{-1}, \quad \alpha = Eha_1^2 D^{-1}, \quad k = a_1 R^{-1}, \quad Q = qDa_1^{-3}$$

$$(D = Eh^3 [12(1-\nu^2)]^{-1})$$

Here W is the panel deflection, F is a force function, E is Young's modulus, ν is Poisson's ratio, h is the panel thickness, R is the radius of curvature in the x_2-direction, and ρ is the density of the material. It is assumed that the transverse load Q is a function of the longitudinal coordinate x_1 and the time τ. The panel planform occupies the rectangle $|x_\beta| \leqslant a_\beta, \beta = 1, 2$. The boundary conditions (1.3) correspond to a free edge, and (1.4) to a fixed hinge support.

Besides problem (1.1)-(1.4), the non-linear integrodifferential equation of the vibrations of a circular arch, written below in dimensionless form

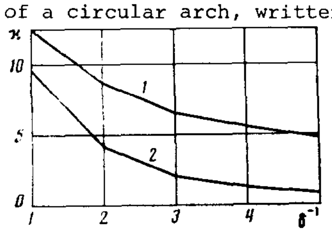

Fig.1

$$(1-\nu^2)\partial_x^4 w + \partial_t^2 w - \frac{\alpha}{2}(k + \partial_x^2 w)\int_{-1}^{1}\left[\frac{1}{2}(\partial_x w)^2 - kw\right]dx = q \tag{1.6}$$

$$[w, \partial_t w]_{t=0} = 0, \quad [w, \partial_x^2 w]_{x=\pm 1} = 0$$

is considered.

2. Construction of the asymptotic expansions. A natural small parameter δ occurs in the system of Eqs.(1.1)-(1.4) as a factor ahead of part of the higher derivatives. Therefore, there is the problem of constructing an asymptotic form as $\delta \to 0$.

Asymptotic expansions are constructed in the form

$$w = \sum_{m=0}^{\infty} \delta^m \left[w_m(x, y, t) + u_m\left(\frac{1+x}{\delta}, y, t\right) + v_m\left(\frac{1-x}{\delta}, y, t\right)\right] \tag{2.1}$$

$$\Phi = \sum_{m=0}^{\infty} \delta^m \left[\Phi_m(x, y, t) + \varphi_m\left(\frac{1+x}{\delta}, y, t\right) + \psi_m\left(\frac{1-x}{\delta}, y, t\right)\right]$$

The functions w_m, Φ_m are found by using a first iteration process /1/. For this the solution is sought in the form

$$\{w, \Phi\} = \sum_{m=0}^{\infty} \delta^m \{w_m, \Phi_m\} \tag{2.2}$$

We substitute (2.2) into (1.1)-(1.4) and collect coefficients of identical powers of δ. Equating the coefficients of δ^0 and δ^1 to zero, to determine w_0, Φ_0 and w_1, Φ_1 we obtain

$$\partial_y^4 w_m = 0; \quad [\partial_y^2 w_m, \partial_y^3 w_m]_{y=\pm 1} = 0; \quad [w_m, \partial_x^2 w_m]_{x=\pm 1} = 0; \tag{2.3}$$

$$m = 0, 1$$

$$\partial_y^4 \Phi_m = 0; \quad [\partial_x^2 \Phi_m, \partial_x \partial_y \Phi_m]_{y=\pm 1} = 0; \quad [\partial_y^2 \Phi_m, \partial_x \partial_y^2 \Phi_m]_{x=\pm 1} = 0$$

Seeking w_m, Φ_m in the form

$$\{w_m, \Phi_m\} = \sum_{j=0}^{3} y^j \{w_{m,j}, \Phi_{m,j}\}$$

we have from (2.3)

$$w_m = w_{m,0}(x,t) + y w_{m,1}(x,t), \quad [w_{m,0}, \partial_x^2 w_{m,0}]_{x=\pm 1} = 0, \quad \Phi_m = 0 \tag{2.4}$$

The function $w_{0,0}$ is still unknown and will be determined below. The function Φ_0 is taken to be equal to zero since it follows from the formulation of the problem (1.1)-(1.4) that the function Φ is determined to the accuracy of linear components in x and y. Continuing the iteration process it is found that the functions $w_{m,j}, \Phi_{m,j}$ vanish for odd values of m and j. Consequently, henceforth in this paper we speak at once about evaluating the function $w_{m,j}, \Phi_{m,j}$ for even m and j.

Equating the expression for δ^2 to zero and taking (2.4) into account, we deduce

$$\begin{aligned} &w_2 = w_{2,0}(x,t) + y^2 w_{2,2}(x,t); \quad 2w_{2,2} = -\nu \partial_x^2 w_{0,0}; \\ &\Phi_2 = C_2(t)\, y^2 \end{aligned} \tag{2.5}$$

The function $w_{2,0}$ is still unknown and will be determined below. At this stage of the first iteration process, the conditions on the boundary $x = \pm 1$ are not satisfied. The discrepancies occurring here are later compensated by using boundary-layer functions.

To determine $C_2(t)$ we will use the well-known identity connecting the functions Φ and w for a fixed reinforcement of the boundary $x = \pm 1$ in the longitudinal direction

$$\int_{-1}^{1} (\partial_y^2 \Phi - \nu \delta^2 \partial_x^2 \Phi)\, dx = \alpha \delta^2 \int_{-1}^{1} \left[\frac{1}{2} (\partial_x w)^2 - k w \right] dx \tag{2.6}$$

Using (2.2) and (2.5), we deduce from (2.6) that

$$C_2(t) = \frac{\alpha}{4} \int_{-1}^{1} \left[\frac{1}{2} (\partial_x w_{0,0})^2 - k w_{0,0} \right] dx \tag{2.7}$$

Equating the expression for δ^4 to zero, we obtain the system of equations

$$\begin{aligned} &\partial_y^4 w_4 + 2\partial_x^2 \partial_y^2 w_2 + \partial_x^4 w_0 + \partial_t^2 w_0 - (k + \partial_x^2 w_{0,0})\, \partial_y^2 \Phi_2 = q \\ &[\partial_y^2 w_4 + \nu \partial_x^2 w_2, \quad \partial_y^3 w_4 + (2-\nu)\, \partial_x^2 \partial_y w_2]_{y=\pm 1} = 0 \\ &\partial_y^4 \Phi_4 + \alpha (k + \partial_x^2 w_{0,0})\, \partial_y^2 w_2 = 0, \quad [\partial_x^2 \Phi_4, \partial_y \partial_x \Phi_4]_{y=\pm 1} = 0 \end{aligned} \tag{2.8}$$

We find from (2.8)

$$\begin{aligned} &w_4 = \sum_{m=0}^{2} y^{2m} w_{4,2m}(x,t), \quad w_{4,4} = \frac{\nu - 2}{12} \partial_x^2 w_{2,2} \\ &w_{4,2} = (1-\nu)\, \partial_x^2 w_{2,2} - \frac{\nu}{2} \partial_x^2 w_{2,0} \end{aligned} \tag{2.9}$$

Here $w_{4,0}$ is also an unknown function. Taking account of (2.5) and (2.7) to determine the principal term of the expansion (2.2) from (2.8) and (2.9), we obtain the integrodifferential Eq.(1.6) for which the zero-th initial and boundary conditions are derived from (1.2) and (1.4) by using (2.4).

Changing to dimensional variables in (1.6) by means of (1.5), we arrive at the well-known equation of arch vibrations. Furthermore, we find Φ_2, the principal term of the expansion (2.2) for the function Φ from (2.5) and (2.7).

Let us now construct the next terms of the asymptotic form. It can be shown that w_{2m}, Φ_{2m} are determined in the form

$$\{w_{2m}, \Phi_{2m}\} = \sum_{j=0}^{m} y^{2j} \{w_{2m,2j}(x,t), \Phi_{2m,2j}(x,t)\}$$

In particular, we have from (2.5), (2.6) and (2.8)

$$\Phi_{4,0} = \Phi_{4,4} = \frac{\alpha \nu f}{24} \partial_x^2 w_{0,0}; \quad \Phi_{4,2} = -2\Phi_{4,4} + \int_{-1}^{1} \Phi_{4,4}\, dx + \frac{g}{2}$$

To determine $w_{2,0}$ we derive

$$(1-\nu^2)\,\partial_x^4 w_{2,0} + \partial_t^2 w_{2,0} - 2\Phi_{2,2}\partial_x^2 w_{2,0} - fg = 2f\int_{-1}^{1}\Phi_{4,4}\,dx + \tag{2.10}$$

$$\frac{\nu}{6}\,\partial_x^2 q + \frac{2}{3}\nu^2(1-\nu)\,\partial_x^6 w_{0,0}, \quad w_{2,0}\,|_{x=\pm1} = 0$$

$$\left(f = k + \partial_x^2 w_{0,0}; \quad g = -\frac{\alpha}{2}\int_{-1}^{1} w_{2,0} f\,dx\right)$$

We note that, unlike (1.6), Eq.(2.10) is linear. Equating the expressions for δ^{2m+2}, δ^{2m+4} $(m = 2, 3, \ldots)$ to determine the functions w_{2m} and Φ_{2m} we obtain

$$(1-\nu^2)\,\partial_x^4 w_{2m,0} + \partial_t^2 w_{2m,0} - 2f\Phi_{2m+2} - 2\Phi_{2,2}\partial_x^2 w_{2m,0} = l_{2m,0} \tag{2.11}$$

$$\Phi_{2m+2,2} = C_{2m+2}(t) - \sum_{j=2}^{m+1} j\Phi_{2m+2,2j}; \quad \Phi_{2m+2,0} = \sum_{j=2}^{m+1}(j-1)\Phi_{2m+2,2j}$$

The functions $l_{2m,0}$, $\Phi_{2m+2,2j}$ $(j = 2, \ldots, m+1)$ are found in the previous stages of the first iteration process, while the functions $w_{2m,2j}$ are calculated in terms of derivatives of the functions $w_{0,0}, w_{2,0}, \ldots, w_{2m-4,0}$. The functions $C_{2m+2}(t)$ $(m > 1)$ are determined from the identity (2.6) on substituting the expansion (2.1).

The boundary-layer functions u_m, φ_m (v_m, ψ_m), concentrated in the neighbourhood of $x = -1$ $(x = 1)$, compensate for the discrepancies in satisfying the boundary conditions (1.4). They are determined by using the second iteration process /1/. Boundary values for $\partial_x^2 w_{2,0}$, $w_{2m,0}$, $\partial_x^2 w_{2m,0}$ $(m \geqslant 2)$, needed to close Eqs.(2.10) and (2.11) are obtained here simultaneously. We substitute (2.1) into (1.1)-(1.4), we take account of the results of the first iteration process, we make the change of variables $x = -1 + \delta\xi$ $(x = 1 + \delta\zeta)$ and we collect coefficient for identical powers of δ. Equating the coefficients for δ^0 to zero, we find a system of non-linear equations with zero right-hand side for u_0, φ_0 from which we obtain $u_0 = \varphi_0 = 0$. Equating the coefficients for $\delta^1, \delta^2, \delta^3$ to zero we deduce

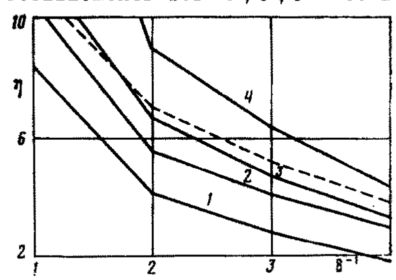

Fig.2

$$u_1 = \varphi_1 = u_2 = u_3 = 0, \quad \Delta_2^2\varphi_2 = 0, \quad [\partial_\xi^2\varphi_2,\ \partial_\xi\partial_y\varphi_2]_{y=\pm1} = 0 \tag{2.12}$$

$$A\varphi_2\,|_{\xi=0} = 2\nu C_2(t), \quad B\varphi_2\,|_{\xi=0} = 0, \ [A\varphi_2, B\varphi_2]_{\xi=l\to\infty} \to 0$$

$$\Delta_2^2 u_4 = k\partial_y^2\varphi_2, \quad [u_4, \partial_\xi^2 u_4]_{\xi=l\to\infty} \to 0 \tag{2.13}$$

$$[\partial_y^2 u_4 + \nu\partial_\xi^2 u_4,\ \partial_y^3 u_4 + (2-\nu)\,\partial_\xi^2\partial_y u_4]_{y=\pm1} = 0$$

$$u_4\,|_{\xi=0} = -w_4\,|_{x=-1}, \quad \partial_\xi^2 u_4\,|_{\xi=0} = -\partial_x^2 w_2\,|_{x=-1}$$

$$(\Delta_2 = \partial_\xi^2 + \partial_y^2, \quad A = \partial_\xi^2 - \nu\partial_y^2,$$

$$B = \partial_\xi^3 + (2+\nu)\,\partial_\xi\partial_y^2, \quad l = 2/\delta)$$

We note that the boundary value problems for u_m, φ_m are linear for $m \geqslant 1$. The functions v_m, ψ_m are found analogously.

We will illustrate the calculation of the boundary-layer function u_4 for the case of a rectangular plate $(k = 0)$. We construct the solution in the form

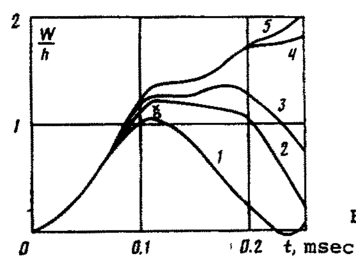

Fig.3

$$u_4 = a_0 e^{-s_0\xi} F_0(y) + 2\,\mathrm{Re}\sum_{m=1}^{\infty} a_m e^{-s_m\xi} F_m(y)$$

The Papkovich functions $F_m(y)$ /7/ are determined from the boundary value problem (the prime denotes the derivative with respect to y)

$$F_m^{\mathrm{IV}} + 2s_m^2 F_m'' + s_m^4 F_m = 0 \tag{2.14}$$

$$[F_m'' + \nu s_m^2 F_m,\ F_m''' + (2-\nu)\,s_m^2 F_m']_{y=\pm1} = 0$$

(s_0, s_m are, respectively, the real and complex roots of the equation $\Psi(s) \equiv (3+\nu)\sin 2s - (1-\nu)\,2s = 0$).

To calculate a_m from the boundary conditions (2.13), the problem is posed of representing the two real functions $f_1 = -w_4(-1, y, t)$ and $f_2 = -\partial_x^2 w_2(-1, y, t)$ in the form of the series

$$\{f_1, f_2\} = \sum_{m=1}^{\infty}\{1, s_m^2\}\,a_m F_m(y) \tag{2.15}$$

Here the time t plays the part of a parameter.

For the first boundary value problem, questions of the completeness of the system of elementary Papkovich solutions for the biharmonic operator in a half-strip were investigated in /8-10/. The conditions for series (2.15) to converge for $F_m(\pm 1) = F_m'(\pm 1) = 0$ are obtained in /11/, and in /12/ for the problem (2.14). By using these conditions we find missing boundary condition for problem (2.10) and one for problem (2.11) for $m = 2$.

To obtain the initial conditions for $t = 0$ for the function $w_{2m,0}$ we substitute (2.1) into (1.2), and we collect coefficients of identical powers of δ and equate them to zero. In particular, the coefficient for δ^0 yields the initial conditions written in (1.6) for $w_{0,0}$. The consistency conditions

$$q(\pm 1, 0) = \partial_x^2 q(\pm 1, 0) = \partial_t q(\pm 1, 0) = 0$$

should be **satisfied here.**

The coefficients of δ^2 and δ^4 reduce, respectively, to the zero-th initial conditions for the functions $w_{2,0}, \partial_t w_{2,0}$ and $w_{4,0}, \partial_t w_{4,0}$. Analogous consistency conditions on the higher derivatives of q are added to construct the next terms of the expansion.

After evaluating the principal terms of expansion (2.1) the process of constructing the next terms of the asymptotic form is continued **analogously: functions of the** first and second iteration processes are determined alternately. The boundary values of the functions of the first iteration process $w_{2m,0}, \partial_x^2 w_{2m,0}$ are determined simultaneously in the solution of the boundary layer problems.

Remark 1°. In the case of rigid clamping of the panel edges $x_1 = \pm a_1$ $([w, \partial_x w]_{x=\pm 1} = 0)$ the principal term of the expansion is also determined from the equation of arch vibrations, but with the boundary conditions $[w, \partial_x w]_{x=\pm 1} = 0$.

2°. In the case of hinge of supports or rigid clamping of the boundaries $x_2 = \pm a_2$ there is no passage to the limit from the equations of the vibrations of a cylindrical panel to the equations of the vibrations of an arch.

3°. The algorithm elucidated can be carried over to the case of a load of the form $Q(x_1, -x_2, \tau) = Q(x_1, x_2, \tau)$.

3. Linear vibrations of a rectangular plate.

Applying the method of Sect.2 to the linear equation of the vibrations of a rectangular plate with two hinge-supported and two free sides

$$\frac{Eh^3}{12(1-\nu^2)} \Delta^2 W + \rho h \partial_\tau^2 W = Q(x_1, \tau), \quad [W, \partial_\tau W]_{\tau=0} = 0 \tag{3.1}$$

$$[\partial_2^2 W + \nu \partial_1^2 W, \partial_2^3 W + (2-\nu)\partial_1^2 \partial_2 W]_{x_2=\pm a_2} = 0;$$

$$[W, \partial_1^2 W]_{x_1=\pm a_1} = 0$$

$$\left(\partial_\tau = \frac{\partial}{\partial \tau};\ \partial_\beta = \frac{\partial}{\partial x_\beta};\ \beta = 1,2;\ \Delta = \partial_1^2 + \partial_2^2\right)$$

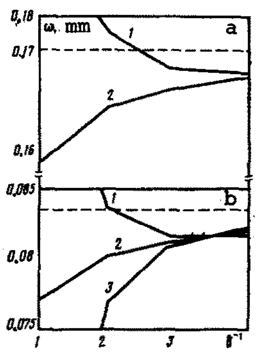

Fig.4

we obtain that the principal term of the asymptotic expansion is determined from the known linear equation of the vibrations of a beam

$$\frac{Eh^3}{12} \partial_1^4 W_0 + \rho h \partial_\tau^2 W_0 = Q \tag{3.2}$$

$$[W_0, \partial_\tau W_0]_{\tau=0} = 0, \quad [W_0, \partial_1^2 W_0]_{x_1=\pm a_1} = 0$$

To estimate the effectiveness of the asymptotic method, problems (3.1) and (3.2) were investigated numerically for a constant step load with amplitude $Q_0 = 5$ kPa, applied instantaneously at the time $\tau = 0$. An explicit finite difference scheme was used for both problems. The geometric and mechanical parameters were assumed to be as follows: $E = 210$ GPa; $\rho = 7850$ kg/m^3; $\nu = 0.3$; $h = 1$ mm; $a_2 = 20$ mm; $\delta = a_2/a_1$. In Fig.1 we show a graph of

$$\varkappa(\delta) = 200 \frac{|1-\gamma|}{|1+\gamma|}, \quad \gamma = \frac{W(0, 0, T_\delta)}{W_0(0, T_0)}$$

where T_δ and T_0 are the times at which the first maximum of the deflection function occurs for plates with parameter δ and beams, respectively. Curve *1* corresponds to rigid clamping and curve *2* to hinge - supports of the edges $x_1 = \pm a_1$. In particular, the value of $\varkappa(0.5)$ equals 4.3% and 8.7%, respectively.

We note that for the problem regarding the equilibrium of a rectangular plate compressed in the longitudinal direction, the problem of replacing it by a compressed rod in the calculations was examined in /13/ (p.255) and /14/ (p.16).

4. Dynamic snap-through of an elastic cylindrical panel.

We will consider the equations non-linear of vibrations of an elastic cylindrical panel, taking inertial terms in the tangential directions into account /6/

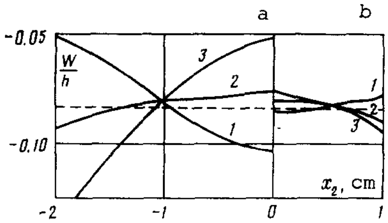

Fig.5

$$\rho h \partial_\tau^2 W = Q + \partial_\alpha (\partial_\beta M_{\alpha\beta} + N_{\alpha\beta}\partial_\beta W) - k_{\alpha\beta}N_{\alpha\beta}; \quad \alpha, \beta = 1,2 \tag{4.1}$$

$$\rho h \partial_\tau^2 U_\alpha = \partial_\beta N_{\alpha\beta}; \quad \{N_{\alpha\beta}, M_{\alpha\beta}\} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\{1, z\}\,dz$$

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22})(1 \rightleftarrows 2), \quad \sigma_{12} = \frac{E}{1+\nu}\varepsilon_{12}$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2}[\partial_\alpha U_\beta + \partial_\beta U_\alpha + \partial_\alpha W \partial_\beta W] + k_{\alpha\beta}W - z\partial_\alpha\partial_\beta W$$

$$[W, \partial_\tau W, U_\alpha, \partial_\tau U_\alpha]_{\tau=0} = 0, \quad [W, M_{11}, U_\alpha]_{x_1=\pm a_1} = 0$$

$$[\partial_2 M_{22} + 2\partial_1 M_{12}, M_{22}, N_{12}, N_{22}]_{x_2=\pm a_2} = 0, \ k_{22} = R^{-1}, \ k_{1\alpha} = 0$$

Here U_α are the tangential displacements of the middle surface and $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are, respectively, the forces and moments. The boundary conditions on $x_1 = \pm a_1$ correspond to fixed hinge support and the boundary conditions on $x_2 = \pm a_2$ to a free edge.

Applying the method of asymptotic integration from Sect.2 to problem (4.1), we obtain the equations of the vibrations of an arch to determine the principal terms of the expansion /15/

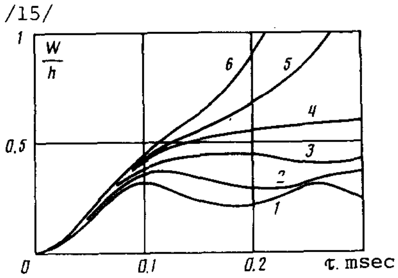

Fig.6

$$\rho h \partial_\tau^2 W_0 = Q + \partial_1(\partial_1 M + N\partial_1 W_0) - \frac{N}{R}; \quad \rho h \partial_\tau^2 U_{1,0} = \partial_1 N \tag{4.2}$$

$$\{N, M\} = \int_{-h/2}^{h/2} \sigma\{1, z\}\,dz, \quad \sigma = E\varepsilon$$

$$\varepsilon = \partial_1 U_{1,0} + \frac{1}{2}(\partial_1 W_0)^2 + \frac{W_0}{R} - z\partial_1^2 W_0$$

$$[W_0, \partial_\tau W_0, U_{1,0}, \partial_\tau U_{1,0}]_{\tau=0} = 0; \quad [W_0, M, U_{1,0}]_{x_1=\pm a_1} = 0$$

To solve problems (4.1) and (4.2), we use an explicit finite difference scheme. Taking account of the symmetry available in problem (4.1), a quarter panel $(x_1, x_2) \in [0, a_1] \times [0, a_2]$ can be considered and the fundamental mesh

$$\left\{(x_{1,i_1}, x_{2,i_2}) \mid x_{\alpha, i_\alpha} = i_\alpha g_\alpha;\ i_\alpha = 0, 1, \ldots, n_\alpha;\ g_\alpha = \frac{a_\alpha}{n_\alpha};\ \alpha = 1, 2\right\}$$

and the auxiliary mesh

$$\left\{(x_{1, i_1-1/2}, x_{2, i_2-1/2}) \mid x_{\alpha, i_\alpha-1/2} \frac{1}{2}(x_{\alpha, i_\alpha} + x_{\alpha, i_\alpha-1});\ i_\alpha = 1, 2, \ldots, n_\alpha\right\}$$

can be introduced on it.

The deflections, strains, stresses, forces and moments will be determined at the nodes of the fundamental mesh while the tangential displacement of the middle surface will be determined at the nodes of the auxiliary mesh.

The following geometrical and mechanical parameters were taken: $a_1 = 20$ mm, $h = 1$ mm, $R = 100$mm, $\rho = 7850\,\mathrm{kg/m^3}$, $E = 210$ GPa and $\nu = 0.3$. Calculations were performed for a panel and arch up to the time $\tau_0 = 300\,\mu$sec with an identical mesh spacing in time. The mesh parameters for problem (4.1) are $n_1 = 2n_2 = 20$. The segment $0 \leqslant x_1 \leqslant a_1$ that was partioned into 20 equal elements was considered in problem (4.2). The quantities

$$\omega(\delta) = \max_{[0,\tau_0]} |W(0, 0, \tau)|, \quad \omega_a = \max_{[0,\tau_0]} |W_0(0, \tau)|, \quad \eta(\delta) = 100\left|\frac{\omega(\delta) - \omega_a}{\omega(\delta)}\right|$$

were determined when calculating the vibrations of a panel for different values of the parameter δ and of an arch.

Here $\omega(\delta)$ and ω_a are the maxima of the deflections of the panel centre and of the arch respectively.

Results of the solution of problems (4.1) and (4.2) for a constant step load $Q(x_1, \tau) = Q_0$ applied instantaneously at the time $\tau = 0$ are shown in Fig.2. Curves *1-4* are drawn for the dependence $\eta(\delta)$, respectively, for the following values of Q_0: 0.5, 2, 3, 3.8 MPa. It is seen that the solution of the arch equation describes the panel vibrations quite well. For instance , for $\delta = 0.5$ and $Q_0 \leqslant 2$ MPa the disagreement in the results does not exceed 5%.

Fig.3 shows graphs of the time dependence of the **panel** deflection W/h at the point $(0, 0)$ and the arch W_0/h for $x_1 = 0$. Curves *1, 2* and *4* refer to a panel with $\delta = 0.5$ for $Q_0 = 3.8$, 4, 4.1 MPa, respectively. Curves *3* and *5* are obtained from an analysis of a panel with $\delta = 0.25$ and on arch for $Q_0 = 4$ MPa. The deflection maximum for a panel with $\delta = 0.25$ and an arch for $Q_0 = 3.8$ MPa is marked by a circle and a cross. On the basis of the Budiansky-Roth criterion /16/, the critical dynamic snap-through Q_d: $Q_d|_{\delta=1} = Q_d(1) = 4.2$ MPa; $Q_d(0.5) = Q_d(0.35) =$

$Q_d(0.25) = 4.1$ MPa were found with up to 0.1 MPa accuracy for panels with different δ. For the arch, $Q_d = 4$ MPa was obtained. We note that the critical dynamic snap-through load is a "stable" characteristics of an elastic system. Despite the fact that an increase in $\eta(\delta)$ (see Fig.2) occurs as the load Q_0 approaches the critical load Q_d while the functions $W(0, 0, \tau)$ and $W_0(0, \tau)$ differ substantially in amplitude and frequency of vibration (see Fig.3), the magnitude of the critical load itself is determined with a 2.5% discrepancy by arch theory.

Results of analyses of panels and arches with rigidly clamped edges $x_1 = \pm a_1$ under the action of a load $Q_0 = 2$ MPa are indicated by dashes in Fig.2. Compared to the case of hinged supports the appropriate values of the function $\eta(\sigma)$ are somewhat higher here. For the critical dynamic snap-through load of a rigidly clamped panel we obtain $Q_d(1) = 4.4$ MPa, $Q_d(0.5) = Q_d(0.35) = Q_d(0.25) = 4.3$ MPa. The critical dynamic snap-through load equals 4.2 MPa for an arch.

The influence of the transverse load distribution over the panel width (the coordinate x_2) can be estimated from Fig.4, where the dependence $\omega(\delta)$ is presented. Cuves *1-3* are obtained from analysis of a panel under the loads $Q_1 = 2Q_0(1 - |x_2|/a_2)$, $Q_2 = Q_0$, $Q_3 = 2Q_0|x_2|/a_2$. The dashed line corresponds to the maximum deflection of an arch ω_a for a load Q_0. Fig.4 shows the results for $Q_0 = 1$ MPa and Fig.4b shows the results for $Q_0 = 0.5$ MPa. It is seen that as the parameter δ decreases the influence of the load distribution non-uniformity on the panel width drops and can already be neglected for $\delta = 0.35$.

Fig.5 shows the distribution of the deflection functions along the line $x_1 = 0$ at the time $\tau = 70$ µsec for a panel with $\delta = 1$ (a) and $\delta = 0.5$ (b) evaluated for the loads Q_1, Q_2, Q_3 (curves *1-3*, respectively) for $Q_0 = 0.5$ MPa. The arch deflection at the point $x_1 = 0$ is superposed by a dashed line. It is seen that a derease in the width increases the panel stiffness in the x_2 direction and the non-uniformity of the application of the load has only a slight effect on the results.

5. Dynamic snap-through of an elasto-plastic cylindrical panel.

Numerical results analogous to those in Sect.4 are obtained in the case of a cylindrical panel with the same geometrical and mechanical parameters, with a yield point $\sigma_T = 0.24$ GPa and tangential modulus $E_T = 2.1$ GPa. The governing relationships were taken according to flow theory with kinematic hardening /17/. The critical dynamic snap-through loads $Q_d(1) = Q_d(0.5) = 1.7$ MPa $Q_d(0.35) = Q_d(0.25) = 1.6$ MPa are obtained to 0.1 MPa accuracy for a step load by application of the Budiansky-Roth criterion. For an arch $Q_d = 1.6$ MPa is obtained.

Fig.6 shows graphs of the arch deflection $W_0(0, \tau)/h$ and panel deflection $W(0, 0, \tau)/h$ for different values of the parameter δ and the load Q_0. Curves *1* and *2* correspond to a calculation with $\delta = 0.5$ for $Q_0 = 1.4$ MPa and $Q_0 = 1.5$ MPa, respectively. Curves *3-5* are obtained for $Q_0 = 1.6$ MPa and correspond to the following values of δ: 1.0, 0.5, 0.25. The graph of the dependence of the arch deflection $W_0(0, \tau)$ for the load $Q_0 = 1.6$ MPa is denoted by the number *6*. It is seen that ignoring the plastic properties of the panel material significantly exaggerates the magnitude of the critical dynamic snap-through load. At the same time the nature of the convergence of the solution for a panel of decreasing width to the solution for an arch is analogous to that obtained in the case of the elastic behaviour of the material.

REFERENCES

1. VISHIK M.I. and LYUSTERNIK L.A., Regular degeneration and boundary layer for linear differential equations with a small parameter, Usp. Matem. Nauk, 12, 5, 1957.
2. GOL'DENVEIZER A.L., Theory of Thin Shells, Nauka, Moscow, 1976.
3. CIAVLET Ph, and RABIER P., The Karman Equations, Mir, Moscow, 1983.
4. SUGIMOTO N., Non-linear theory for flexural motions of thin elastic plates, Trans. ASME, J. Appl. Mech., 48, 2, 1981.
5. KUCHERENKO V.V. and POPOV V.A., Asymptotic behaviour of the solutions of problems in the theory of elasticity in thin domains, Dokl. Akad. Nauk SSSR, 274, 1, 1984.
6. VOL'MIR A.S., Non-linear Dynamics of Plates and Shells, Nauka, Moscow, 1972.
7. PAPKOVICH P.F., On a mode of the solution of the plane problem of the theory of elasticity for a rectangular strip, Dokl. Akad. Nauk SSSR, 27, 4, 1940.
8. VOROVICH I.I., Some mathematical problems of plate and shell theory. Proceedings of the Second All-Union Congress on Theoret. and Appl. Mech., Nauka, Moscow, 1966.
9. VOROVICH I.I. and KOVAL'CHUK V.E., The basis properties of a system of homogeneous equations, PMM, 31, 5, 1967.
10. USTINOV YU.A. and YUDOVICH V.I., The completeness of a system of elementary solutions of the biharmonic equation in a half-strip, PMM, 37, 4, 1973.
11. GRINBERG G.A., On a method proposed by P.F. Papkovich for solving the plane problem of the theory of elasticity for a rectangular domain and the problem of the bending of a rectangular thin slab with two clamped edges and some of its generalzations, PMM, 17, 2, 1953.

12. STOLYAR A.M., Asymptotic ingegration of the equation of the vibrations of a long rectangular plate, Izv. Sev.-Kavk. Nauchn. Tsentr Vyssh. Shk., Estestv. Nauka, 4, 1986.
13. DONNELL L.G., Beams, Plates and Shells, Nauka, Moscow, 1982.
14. ISHLINSKII A.YU., Applied Problems in Mechanics, 2, Mechanics of Elastic and Absolutely Solid Bodies, Nauka, Moscow, 1986.
15. STOLYAR A.M. and TSIBULIN V.G., Asymptotic integration of the equation of the vibrations of cylindrical panels, Proceedings of the All-Union Symposium on "Non-linear Theory of Thin-Walled Structures and Biomechanics". Izd. Tbilisi Univ., Kutaisi, 1985.
16. BUDIANSKY B. and ROTH R.S., Axisymmetric dynamic buckling of clamped shallow spherical shells, TND-1510, NASA, 1962.
17. ISHLINSKII A.YU., General theory of plasticity with linear hardening, Ukr. Mat. Zh., 6, 3, 1954.

Translated by **M.D.F.**

PMM U.S.S.R.,Vol.52,No.4,pp.518-525,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

THE METHOD OF ASYMPTOTIC INTEGRATION AND THE "METHOD OF SPRINGS" IN PROBLEMS OF ELASTIC PLATES WITH AN ELONGATED CUT*

R.V. GOL'DSHTEIN and L.B. KOREL'SHTEIN

A class of problems in the theory of the elasticity of plates with an elongated non-through cut under arbitrary loading is analysed by the method of asymptotic integration /1-3/. An asymptotic solution in a small parameter (the ratio of the plate thickness and the length of the cut is constructed as the sum of an external solution corresponding to the two-dimensional problem of plate theory and an internal solution corresponding to the boundary layers in a zone of order h near the cut as well as the plate boundaries.

It is shown that the cut affects the elastic state of deformation in the plate (outside the boundary layers) in the second term of the external solution resulting in jumps in the kinematic and force factors on the line of the cut. Equations are obtained that express the jumps mentioned in term of the geometrical parameters of the cut and the energy characteristics of the first terms of the internal solution that is the state of plane and antiplane strain of a strip with the cut under the action of loads on the surface of the cut governed by the forces and moments of the first term of the external solution on the line of the cut. After the solution of the appropriate plane and antiplane problems for the first term of the internal solution, determination of the second term of the external solution reduces thereby to a problem in the theory of plates with the boundary conditions on the line of the cut and the edges of the plate. The second term of the asymptotic form of the boundary layer near the cut is the solution of more complex plane and antiplane problems for a strip with a cut, with a load including volume and surface forces associated with the change in the first term of the solution for the boundary layer along the cut.

Starting from the equation obtained in the case of a cut that is an extended rectilinear surface crack (normal to the plate surface) for both symmetric and antisymmetric loading approximate boundary conditions can be formulated on the line of the crack for a binomial asymptotic form of the external solution, which enables us to pose a problem in the theory of plates taking the influence of cracks into account. For symmetric loading these boundary conditions reduce to equations of the known Rice-

*Prikl.Matem.Mekhan.,52,4,666-674,1988